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Calculus of Variation and its Applications

Rekha Devi
Government Degree College Jhandutta, Bilaspur (H.P.)-174031, India
E-mail: rekha111179@gmail.com

ABSTRACT: The calculus of variation is an efficient technique for the solution of problems in dynamics of rigid bodies and other mathematical problems. The calculus of variation is concerned with the problem of extrmising "functional". This problem is a generalization of the problem of finding extrema of functions of several variables. The calculus of variations gives us precise analytical techniques to find the shortest path (i.e. geodesic) between two given points on a surface. It also used to find the curve between two given points in the plane that yields a surface of revolution around a given axis. It also formulates "Branchistochrone problem. Calculus of variation also underpins much of modern mathematical physics via Hamilton Principle of least action. It can be used both to generate interesting differential equations. The Fermat's Principle in optics and the the principle of least action in physics are written as variational principles.

Keywords: Functionals; geodesic; extrema and branchstochrone.

INTRODUCTION: The basic ideas in the calculus of variations are

(1) Finding the maximum and minimum for a given smooth function$f(x)$: Consider the Taylor expansion of the smooth function$f(x)$.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + f''\big((x_0)(x - x_0\big)2 + \text{ Higher order terms}$$

$f'(x_,)$ >o implies the function $f(x)$is increasing around (x_0)

$f'^{(x)} < 0$ implies the function $f(x)$ is decreasing around x_0

$f'(x) <= 0$ and $f''(x_0) > 0$ implies x_0 is the local minimum

$f(x) > 0$ and $f''(x) < 0$, x_0 is the local maximum.

(2) Formula of Branchistochrone Problem: The Branchistochrone means the "shortest time in Greek. Start the two balls at the top at the same time the one rolling along the curved path travels further but reaches the bottom first. Formulation of the Branchistochrone problem The time of travel from a point P_1 and P_2 is given by

Time $= \int_{p1}^{p2} \frac{ds}{v}$, here s is the arc length and v is the speed

$\frac{1}{2}mv^2 = mg, yv = \sqrt{2gy}$

$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (\frac{dy}{dx})2dx}$

$F(y) = \int_{p1}^{p2} \sqrt{1 + (\frac{dy}{dx})2dx} \,/\, \sqrt{2gy}$

Functional F as a Function defined on a class of Paths $\big(x,_{f(x)}\big)$ which connect two points $p_{1and}\ p_2$

GEODESICES PROBLEMS:

Geodesics Problem in a plane: The Problem of finding the shortest path between two points in the plane. A straight line joining two points P and Q. By a path between P and Q we mean a twice continuously differential curve

$$x\ :\ [0,1] \to R^2, t \to (x''(t), x^2(t))$$

With condition that $x(0) = $ P and $x(1) = Q$ the arc length of such a path is obtained by integrating the norm of the velocity vector.

$$S(x) = \int_0^1 [(x(t))]\, dt = \int_0^1 \|\dot{x}\|\, dt$$

$$\|\dot{x}\| = \sqrt{(\dot{x}_1(t))^2 \dot{+} (\dot{x}_2(t))^2}$$

$$\frac{d}{dt}\left(\frac{\dot{x}}{\|\dot{x}\|}\right) = 0$$

The velocity vector $\dot{x}$ has constant direction i.e that it is a straight line joining P and Q.

Geodesic Problem on a sphere: Let P and Q be any two points on the units sphere S^2 in R^3.

In Spherical polar Co-ordinates

$$x(t) = (\cos\theta(t)\sin\emptyset(t), \sin\theta\sin\emptyset(t), \cos\emptyset(t)$$

The arc Length is computes by integrating $\|\dot{x}\|$

$$\left\|\dot{x}\right\| = \sqrt{\dot{\emptyset}^2 + (\sin\emptyset)^2\dot{\theta}^2}$$

There are length of the path defines a functional on function θ and $\emptyset$

$S[\theta,\emptyset] = \int_0^1 \sqrt{\emptyset^2 + (\sin\emptyset)^2\,\dot{\theta}^2}\ d\emptyset$ The shortest path between P and Q can be found by exteremising the above functional. The arc length functional becomes

$$S[\theta,\emptyset] = \int_{\emptyset_P}^{\emptyset_Q} \sqrt{1 + (\sin\emptyset)^2\,\dot{\theta}^2}\ d\emptyset$$

Application of Calculus of Variation in physics Principle of least action: In Physics, The Principle of least Action states that the motion of a mechanical system will follow the trajectory which minimize the action of the system are called physical trajectories.consider a particle of mass in moving in R^3 under the influence of a potential $V : R^3 \to R$

Let $x : R \to R^3$ denote the trajectory of this particle. Define the Kinetic energy of the trajectory to the function $T : R^3 \to R$ Defined by

$$T(x) = \frac{1}{2}\ m\|\dot{x}\|^2$$

$$L(x,\dot{x}) = T(\dot{x}) - v(x)$$

The action of the trajectory from time t_0 to time t_1 is the integral

$$S[x] = \int_{t_0}^{t1} L\ (x(t),\dot{x}(t)dt$$

Classical Mechanics:-

Consider the one-particle system in a conservative field. The relation between the Potential energy of the particle and the force acting on the particle is given by $F = (F_{1,} F_{2,} F_{3,})$

$$\nabla V = -(V_x, V_y\ V_Z)$$

$$Work = -V(B) + V(A) = \int_A^B F\cdot ds\ \cdot$$

The Kinetic energy of the Particle is

$T = \frac{1}{2}\ m\ (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\cdot$

The associated Lagrangian is

$$L(x,y,z,\dot{x},\dot{y},\dot{z}) = T - V = \frac{1}{2}\ m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x,y,z)$$

Consider the Variational Problem min $S(x(t), y\ (t), z\ (t)) = min\int_0^T \frac{1}{2} m\ (\dot{x}^2 + \dot{y}^2 + z2 - Vx,y,zdt$

$$\delta S = \int_0^T -m\ (\dot{x}\,\delta\ x + \dot{y}\,\delta\ y + z\,\delta\ z - Vx\ \delta x + Vy\,\delta y + Vz\ \delta\ z\,dt$$

$$\delta S = \int_0^T -m\ (\ddot{x}\,\delta\,x + \ddot{y}\,\delta\,y + z\,\delta z + F1\,\delta x + F2\,\delta y + F3\,\delta z\,dt$$

$$\delta S = \int_0^T (F_1 - m\ddot{x})\,\delta x + (F_2 - m\ddot{y})\delta y + (F_{3\,-m\ddot{z})}\ \delta\ddot{z}\ dz$$

According to the Principle of Least action $\delta s = 0$

$$\begin{cases} m\ddot{x} = F_1 \\ m\ddot{y} = F_2 \\ m\ddot{z} = F_3 \end{cases}$$

This is the Newton Second Law of motion, $F = ma\cdot$

Lagrangian Mechanics: The Lagrangian is given by

$$L(x_1\ x_{1\,,}\ x_2, x_{3\,,} \ldots\ldots x_n, \dot{x_{1,}}\ \dot{x_2},\ x_3\dot{\ldots\ldots}, \dot{x}_n) = T[\,\dot{x}_1, \dot{x}_2\ , \ldots\ldots\ldots \dot{x}_n\,) - V(x_1, x_2, x_3 \ldots . x_n,)(\dot{x}_{1,}\dot{x}_2\ , \ldots\ldots\ldots \dot{x}_n\,)$$

where T is the K.E and V is the Potential energy of the system and $(x_{1,} x_2\ , x_3,)$ is a generalised co-ordinate. The action S is defined by

$$S(x_1 x_2 - - - - x_n) = \int_0^T L\ (x_1\ x_{1\,,}\ x_2, x_{3\,,} \ldots\ldots x_n, \dot{x_{1,}}\ \dot{x_2},\ x_3\dot{\ldots\ldots}, \dot{x}_n)\ dt$$

According to the Principle of Least action, equations are:

$$\begin{bmatrix} \dfrac{\partial L}{\partial x_1} & - & \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x}1} = 0 \\ \vdots & = & \vdots \\ \dfrac{\partial L}{\partial x_n} & - & \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{x_n}} = 0 \end{bmatrix}$$

$F \;= (\frac{\partial L}{\partial x_1}\ \frac{\partial L}{\partial L_2} - - - - - - \frac{\partial L}{\partial x_n})$ is a generalized force

$P \;= (\frac{\partial L}{\partial \dot{x}_1}\ \frac{\partial L}{\partial L_2} - - - - - - \frac{\partial L}{\partial x_n})$ is a generalized momentum

Principle of least action

$min\, S\left(x_1 x_2 - - - - x_n\right) \;= \min \int_0^I L\,(x_1 x_2 - - - - xn, \dot{x}_1\ \dot{x}_2 - - - \dot{x}n\Big)\,dt$

Provide a way to find a suitable co-ordinate system for mechanical system.

Hamiltonian Mechanics: The Lagrangian of a system is given by L(q,$\dot{q}$) introduce the Hamiltonian function, $H(q,P) = \ \max_{\dot{q}}(P,) - L(q,\dot{q})$). This is equivalent to $H(q,P) = P\dot{q} - L(q,\dot{q})$ and $P = \frac{\partial L}{\partial \dot{q}}$(q,$\dot{q}$) $w$here the second equation gives us the relation $\dot{q} = \dot{q}\ (q,P)$. Since the double Legendre transformation is itself, we have

$$L(q,\dot{q}) = \text{P}\,\dot{\text{q}} - H(q,p)\ and\ \dot{q}\ = \frac{\partial H}{\partial P}(\text{q,p}).$$

Example: A system of a single particle

$$L(\text{q},\dot{q}) = \frac{m}{2}\,\dot{q}^2 - V(q). \qquad P = \frac{\partial L}{\partial \dot{q}} = m\dot{q}\ ,\dot{q} = \frac{p}{m}$$

$$H(q,P) = P\dot{q} - L\left(q,\dot{q}\ \ = \frac{p^2}{m} - \frac{p^2}{2m} - V(q)\right) = \frac{p^2}{2m} + V\,(q) = Total\ energy$$

Hamilton's equations

Applying the principle of least action to $(L(\text{q},\dot{q}) = \text{P}\ \dot{q} - H(q,p)dt$. we have

$$\delta\int_{t1}^{t2} L\ dt = \delta\int_{t1}^{t2}[p\dot{q} - H(q,p)]\,dt\,.$$

$$\int_{t1}^{t2}\left[\ \dot{q}\ \delta p + p\delta\dot{q} - \frac{\partial \dot{H}}{\partial p}\ \delta p - \frac{\partial H}{\partial q} - \delta q\ \right] dt\ = 0\,,$$

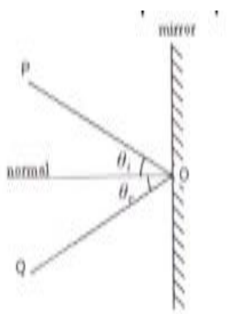
$$\int_{t1}^{t2}\left(\dot{q} - \frac{\partial H}{\partial q}\right)\delta p\ + \left(-\dot{p} - \frac{\partial H}{\partial q}\right)\delta q\ dt\ = 0.$$

This leads us to Hamilton's equations

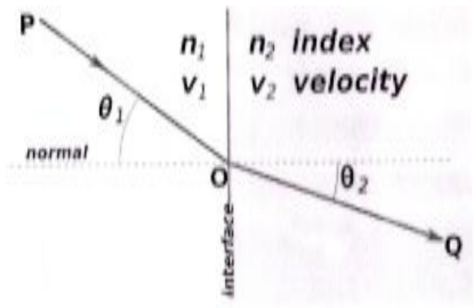
$$\begin{cases} \dot{q} & = -\dfrac{\partial H}{\partial P} \\ \dot{p} & = -\dfrac{\partial H}{\partial q} \end{cases}$$

Fermat's Principle

In optics, the Fermat's Principle , also called the Principle of the least time, states that a path taken by a ray of light between two points is the least-time path among all "possible" paths .The law of reflection and the law of refraction could be derived from the Fermat's Principle.



Reflection



Refraction

The Fermat's Principle motivates the Principle of least action in mechanical systems.

CONCLUSIONS: Calculus of variations seeks to first the path, curve surface etc for which a given function has a stationary value in calculus of variation. We study powerful techniques of solving problems of optimizations of functional .Calculus of variations help to formulate Geodesic problems on a plane and sphere. There are many laws of Physics which are written as variational principles. The Principle of Least action is equivalent to Newton Second Law of motion in a mechanical system. The Fermat Principle in optics is also written as variational principle it also help to formulates Brachistchrone problem.

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